



Fermi National Accelerator Laboratory

FERMILAB-PUB-88/124-T

September, 1988

**On The Determination of Phases of Generalized Kobayashi-Maskawa
Matrix Elements From Their Magnitudes**

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ABSTRACT

For an N-generation Kobayashi-Maskawa mixing matrix having the property that matrix elements decrease rapidly with increasing generation change, we show by construction that knowledge of the moduli of all elements is sufficient to determine all phase information.



There has recently been interest in the fact that knowledge of only the magnitudes of the elements of the 3-generation Kobayashi-Maskawa (K-M) mixing-matrix elements is sufficient to determine all phase information.¹ This implies that, in principle, decay-width information alone can determine the presence or absence of CP-violating couplings. (This inference of course assumes the correctness of the K-M description, in particular that the mixing matrix is unitary).

It is natural to ask whether this feature can be generalized to the case of N generations. A general argument appears at best complicated;¹ the case of four generations has, by a brute force calculation, recently been argued to be correct.² However, counterexamples also have been shown to exist.³

Our goal here is more modest but, hopefully, practical. It is likely that a generalized $N \times N$ K-M matrix will share the property seen for 3 generations, namely that the matrix-elements rapidly get smaller as the generation-change increases. We shall show by construction that this restriction alone allows one to reconstruct the phase information from the magnitudes of the elements.

We assume that all $|V_{ij}|$ are known and that, if $j \geq i$, then

$$|V_{ji}| \gg |V_{j+1,i}| \quad (1)$$

$$|V_{ji}| \gg |V_{j,i-1}| \quad (2)$$

$$|V_{ij}| \gg |V_{i,j+1}| \quad (3)$$

$$\text{and } |V_{ij}| \gg |V_{i-1,j}| \quad (4)$$

This implies that $|V_{ii}| \cong 1$ because all off-diagonal elements are small. We also assume that quark phases are chosen ⁴ such that V_{ii} and $V_{i,i+1}$ are real and nonnegative.

Then the argument proceeds by induction. Suppose all elements V_{ij} for $|i-j| < n$

have been determined. Now for $j - i = n$ use the unitarity restriction

$$\sum_{k=1}^N V_{ik} V_{jk}^* = 0 \quad (5)$$

From this sum, there are two special terms $k = i$ and $k = j$ which contribute, to good approximation, the quantity $V_{ij} + V_{ji}^*$. For $k < i$ and $k > j$ it is easy to see that relative to $|V_{ij}|$ or $|V_{ji}^*|$ the individual magnitudes of the terms are very small; e.g.

$$|V_{ij}| \gg |V_{i,i-1} V_{j,i-1}^*| \gg |V_{i,i-2} V_{j,i-2}^*| \gg \dots \quad (6)$$

Thus the significant terms in the sum come from $i \leq k \leq j$ and we get the main result:

$$V_{ij} + V_{ji}^* \cong -\sum_{k=i+1}^{j-1} V_{ik} V_{jk}^* \quad (7)$$

The right-hand side, by induction, is already determined, so that a triangle construction in the complex plane can be used. Since the lengths of the sides of the triangle and the orientation of the base (the right-hand side) are known, all phase angles are determined.

Some special cases are worth noting:

1) $j = i+1$:

In this case there is no sum on the right-hand side, so that

$$V_{i+1,i}^* \cong -V_{i,i+1} = -V_{i,i+1}^* \quad (8)$$

This gives, as is very well-known

$$\begin{aligned} V_{cd} &\approx -V_{us} \\ V_{ts} &\approx -V_{cb} \end{aligned} \quad (9)$$

and for four generations, in obvious notation

$$V_{Tb} \approx -V_{tB} \quad (10)$$

2) j=i+2:

The one interesting case for 3 generations is the well-known triangle relation⁵

$$(V_{ub} + V_{td}^*) \cong -V_{us}V_{ts} \cong V_{us}V_{cb} \quad (11)$$

For four generations there is, in addition, the relation

$$V_{cB} + V_{Ts}^* \cong -V_{cb}V_{Tb} \quad (12)$$

3) j = i+3:

This first occurs for four generations and leads in that case to one constraint, sufficient to complete the determination of the matrix elements:

$$V_{uB} + V_{Td}^* \cong -(V_{us}V_{Ts}^* + V_{ub}V_{Tb}^*) \quad (13)$$

For the first time, the right hand side becomes a complex vector, leading to a generalized orientation of the unitarity triangle.

It is unclear to this author what range of validity is applicable to this construction. Within our assumptions a perturbative scheme can most certainly be used to refine the formulae given above. But its radius of convergence has not been investigated, nor the detailed connection to the counterexamples found in Ref. 3. Whatever the connection, we may see that the construction we have presented would be in trouble were the right-hand side of Eqn. (7) ever “accidentally” small.

It is interesting to speculate on the pattern of magnitudes of 4-generation K-M matrices consistent with our input assumptions and see whether the unitarity

triangles become nontrivial, i.e. nondegenerate, in the level of approximation we have considered. For example, suppose

$$V_{KM} \sim \begin{pmatrix} 1 & \theta & \theta^3 & \theta^4 \\ \theta & 1 & \theta^2 & \theta^3 \\ \theta^3 & \theta^2 & 1 & \theta \\ \theta^4 & \theta^3 & \theta & 1 \end{pmatrix}, \quad (14)$$

where the entries θ^n indicate only the order of magnitude of the elements, as expressed in terms of powers of the Cabibbo angle. Then we immediately see that all three unitarity-triangle relations (Eqns. 11, 12, and 13) remain nontrivial, the sides being of order θ^3 , θ^3 , and θ^4 respectively. On the other hand, were we to have

$$V_{KM} \sim \begin{pmatrix} 1 & \theta & \theta^3 & \theta^5 \\ \theta & 1 & \theta^2 & \theta^4 \\ \theta^3 & \theta^2 & 1 & \theta^3 \\ \theta^5 & \theta^4 & \theta^3 & 1 \end{pmatrix} \quad (15)$$

then the first and third unitarity triangles would be nontrivial (of size $\sim \theta^3$ and $\sim \theta^5$ respectively, while the second “triangle”, eqn. 12, becomes approximately degenerate.

I thank Carl Albright and Charif Hamzaoui for helpful discussions and suggestions.

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